OTDE-Workshop on functional analysis, 2023-II

The Lindenstrauss Problem and Boolean Valued Analysis

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the Vladikavkaz Scientific Center of
the Russian Academy of Sciences

October 25-27, 2023

Contents

- The Lindenstrauss Problem
- Injective Banach Lattices and Their Preduals
- Boolean Valued Transfer Principle

- This talk is based on the recent joint paper:

  A. G. Kusraev and S. S. Kutateladze, Geometric characterization of preduals of injective Banach lattices, Indag. Math., **31**:5 (2020), 863-878.

- $\mathbb{N}$: the set of all integers $n \geq 1$.

  $\mathbb{R}$: the set of all real numbers.

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I. The Lindenstrauss Problem

L^1-preduals: Definition

- **Definition.** A Banach space X whose dual X' is isometrically isomorphic to $L^1(\mu)$ for some positive measure μ is called an L^1-predual space or a Lindenstrauss spaces.
- The Lindenstrauss Problem:

  Classify and characterize the L^1-predual Banach spaces.
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- Grothendieck conjectured that a Banach space is an L^1-predual iff it is isometric to a subspace of $C(K)$ of the form:

$$\{f \in C(K) : f(k_\alpha^1) = \lambda_\alpha f(k_\alpha^2);\ k_\alpha^1, k_\alpha^2 \in K; \lambda_\alpha \in \mathbb{R};\ \alpha \in \mathrm{A}\}.$$

- **Definition.** A BS X is called a Grothendieck space (or G-spaces) if it admits the above functional representation.
- A G-space is an L^1-predual. The converse is false as demonstrated by Lindenstrauss in his memoir.

  The Grothendieck conjecture is true for spaces X with $\operatorname{ext} B(X')$ w^*-compact.

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- **Theorem (Lindenstrauss, 1964).** For a Banach space X the following assertions are equivalent:
  (1) X is an L^1-predual space.

  (2) Every family of 4 pairwise intersecting closed balls in X has a non-empty intersection.

  (3) Every compact operator $T : Y \to X$ has, for every $\varepsilon > 0$ and Banach spaces Y, Z, $Z \supset Y$, a compact extension $\hat{T} : Z \to X$ with $\|\hat{T}\| \leq (1 + \varepsilon)\|T\|$.
- **In complex case:**
  (2′) Every family of 4 balls in X such that any 3 of them have a non-empty intersection, has a non-empty intersection.
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Chebyshev Radius and Centerable Sets

- **Definition.** Let X be a Banach space and $A \subset X$. The *diameter* $\delta(A)$ of A is defined as

$$\delta(A) = \sup\{\|a - b\| : a, b \in A\}.$$

- **Definition.** The *Chebyshev radius* $r(A)$ of A is defined as

$$r(A) = \inf_{x \in X} r(A, x); \quad r(A, x) = \sup_{a \in A} \|x - a\| \quad (x \in X).$$

- It is easily seen that $\delta(A) \leq 2r(A)$.
- **Definition.** If $\delta(A) = 2r(A)$, then A is said to be *centerable*.
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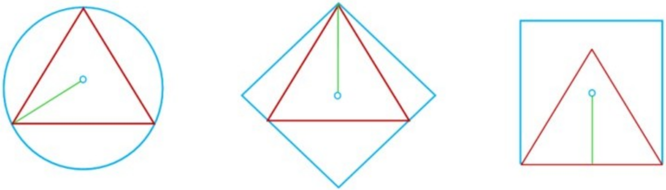
$\triangle$ is an equilateral triangle with side length 1;

$\delta(\triangle)$ is the diameter of $\triangle$;

$r(\triangle)$ is the Chebyshev radius of $\triangle$.

$\|(x,y)\|_2 := \sqrt{x^2+y^2}$; $\|(x,y)\|_\infty := \max\{|x|,|y|\}$.

$\|(x,y)\|_1 := |x|+|y|$;



$(\mathbb{R}^2, \|\cdot\|_2)$ $\quad$ $(\mathbb{R}^2, \|\cdot\|_1)$ $\quad$ $(\mathbb{R}^2, \|\cdot\|_\infty)$

$\delta(\triangle) = 1 < \frac{2}{\sqrt{3}} = 2r(\triangle)$

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- **Definition.** The relative Chebyshev radius of A (w.r. $B \subset X$):
$$r_B(A) = \inf\{r(A, x) : \; x \in B\}.$$

- **Theorem (Garkavi, 1964; Klee, 1968).** For a normed space X the following assertions are equivalent:

  (1) X is an inner product space.

  (2) $r_Y(A) = r_X(A)$ for every 2-dimensional subspace Y of X and every bounded $A \subset Y$.

  (3) $r_Y(\Delta) = r_X(\Delta)$ for every 2-dimensional subspaee Y of X and every triplet $\Delta = \{y_1, y_2, y_3\} \subset Y$.

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Characterization of Inner Product Spaces

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- **Theorem.** For a real BS X the following are equivalent:

  (1) X is an L^1-predual space.

  (2) Every four-point subset of X is centerable.

  (3) Every finite subset of X is centerable.

  (4) Every compact subset of X is centerable.
- **Remarks:**

  ✓ The result is true also for complex Banach spaces.

  ✓ (1) $\iff$ (3) is due to T. S. S. R. K Rao (2002).

  ✓ This result cannot be sharpened anymore: i.e., the centerability of every three-point subset of a real or complex Banach space X does not imply that X is an L^1-predual space.

Characterization of L^1-preduals

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II. Injective Banach Lattices and Their Preduals

- **Definition.** A *vector lattice* (VL for short) is a real vector space X equipped with a partial order $\leq$ for which there exist

  ✓ $x \vee y := \sup\{x, y\}$, the supremum,

  ✓ $x \wedge y := \inf\{x, y\}$, the infimum,

  for all vectors $x, y \in X$ and such that the *positive cone*

  ✓ $X_+ := \{x \in X : \ x \geq 0\}$ of X have the properties

  ✓ $X_+ + X_+ \subset X_+, \quad \mathbb{R}_+ \cdot X_+ \subset X_+$.

- **Definition.** A *Banach lattice* (BL for short) is a Banach space which is also a VL and the order is connected to the norm by

  ✓ $|x| \leq |y| \implies \|x\| \leq \|y\|$ (*monotonicity*),

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Banach Lattices

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- **Definition.** A Banach lattice X is called *AM-space* if

$$\|x \vee y\| = \max\{\|x\|, \|y\|\} \text{ for all positive } x, y \in X.$$

- **Example.** The space $C(K)$ of continuous functions on a compact Hausdorff space K with the supremum norm

$$\|f\|_\infty := \sup\{|f(t)| : \ t \in K\} \ (f \in C(K)).$$

$C(K)$ is order complete iff K is extremally disconnected.

- **Definition.** $0 \leq \mathbb{1} \in X$ is a *strong order unit* if $0 \in \mathrm{int}[-\mathbb{1}, \mathbb{1}]$ where $[-\mathbb{1}, \mathbb{1}] := \{x \in X : \ -\mathbb{1} \leq x \leq \mathbb{1}\}$ is an *order interval*.
- **Theorem (Br. Kreĭns–Kakutani, 1941).** An arbitrary *AM*-space with strong order unit is lattice isometric to $C(K)$ for some compact Hausdorff space K.

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- **Definition.** A Banach lattice X is an *AL-space*, provided

$$\|x + y\| = \|x\| + \|y\| \text{ for all positive } x, y \in X.$$

- **Example.** Given a measure space (Ω, Σ, μ), denote by $L^0(\Omega, \Sigma, \mu)$, $L^1(\Omega, \Sigma, \mu)$, and $L^\infty(\Omega, \Sigma, \mu)$ respectively the vector lattice of (classes of equivalence of) all measurable integrable, essentially bounded functions on Ω.

  Evidently, $L^1(\mu) := L^1(\Omega, \Sigma, \mu)$ is an AL-space;

  $$L^\infty(\mu) := L^\infty(\Omega, \Sigma, \mu) \text{ is an } AM\text{-space.}$$

- **Theorem (Kakutani, 1941).** An AL-space X is lattice isometric to $L^1(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ).
- **Remark.** Any of the vector lattices $L^1(\Omega, \Sigma, \mu)$, $L^0(\Omega, \Sigma, \mu)$, and $L^\infty(\Omega, \Sigma, \mu)$ is order complete iff (Ω, Σ, μ) is localizable.

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  Evidently, $L^1(\mu) := L^1(\Omega, \Sigma, \mu)$ is an AL-space;

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- **Theorem (Kakutani, 1941).** An AL-space X is lattice isometric to $L^1(\Omega, \Sigma, \mu)$ for some measure space (Ω, Σ, μ).

- **Remark.** Any of the vector lattices $L^1(\Omega, \Sigma, \mu)$, $L^0(\Omega, \Sigma, \mu)$, and $L^\infty(\Omega, \Sigma, \mu)$ is order complete iff (Ω, Σ, μ) is localizable.

Banach Lattices: AL-spaces

- **Definition.** A Banach lattice X is an *AL-space*, provided
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- **Example.** Given a measure space (Ω, Σ, μ), denote by $L^0(\Omega, \Sigma, \mu)$, $L^1(\Omega, \Sigma, \mu)$, and $L^\infty(\Omega, \Sigma, \mu)$ respectively the vector lattice of (classes of equivalence of) all measurable integrable, essentially bounded functions on Ω.
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- **Definition.** An **injective Banach lattice** is a real BL X:

$$(\forall Y)\,(\forall Y_0)\,(\forall T_0)$$

$$\left.\begin{matrix} Y_0, Y \in (\mathrm{BL}) \\ Y_0 \text{ is a closed sublattice of } Y \\ 0 \le T_0 \in L(Y_0, X) \end{matrix}\right] \Longrightarrow \left[\begin{matrix} (\exists T) \\ 0 \le T \in L(Y, X) \\ T|_{X_0} = T_0 \\ \|T\| = \|T_0\| \end{matrix}\right.$$

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III. Boolean Valued Transfer Principle

What Is Boolean Valued Analysis?

- Boolean valued analysis is a branch of functional analysis which uses a special model-theoretic technique and consists in studying the properties of a mathematical object by means of comparison between its representations in two different set-theoretic models whose construction utilizes distinct Boolean algebras.
- The von Neumann universe (Cantorian paradise) $\mathbb{V}$ and a specially-trimmed Boolean valued universe $\mathbb{V}^{(\mathbb{B})}$ are taken as these models.
- The comparative analysis requires some ascending–descending machinery to carry out the interplay between $\mathbb{V}$ and $\mathbb{V}^{(\mathbb{B})}$.

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$L^0(\mu)$:= $L^0(\Omega, \Sigma, \mu)$, the space (of classes) of integrable functions with respect to a measure $\mu : \Sigma \to \mathbb{R}$.

$\mathbb{B}$:= $\Sigma/\mu^{-1}(0)$ Boolean algebra of all measurable sets modulo μ-null sets.

$[\![\varphi]\!] \in \mathbb{B}$ is the Boolean truth value of ZFC formuls φ

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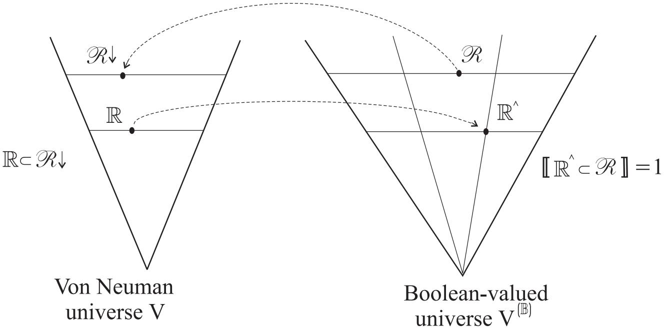

- **Theorem (Kusraev, 2012).** *Every IBL embeds into an appropriate Boolean-valued model, becoming an AL-space.*
- **A Transfer Principle.** Each theorem about the AL-space within Zermelo–Fraenkel set theory has its counterpart for the original injective Banach lattice interpreted as a Boolean-valued AL-space.
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L-Projections and M-Projections

- **Definition.** A projection π on a Banach space X is said to be: an M-projection if $\|x\| = \|\pi x\| + \|x - \pi x\|$ for all $x \in X$; an L-projection if $\|x\| = \max\{\|\pi x\|, \|x - \pi x\|\}$ for all $x \in X$.
- **Notation.** $\mathbb{P}_L(X)$ and $\mathbb{P}_M(X)$ denote the sets of all L-projections and M-projections on X.
- A Boolean algebra of projections on X is a commuting set $\mathbb{B}$ of linear norm one projections in X with:

$$\pi \wedge \rho := \pi \circ \rho, \quad \pi \vee \rho := \pi + \rho - \pi \circ \rho, \quad \pi^{\perp} := I_X - \pi.$$

- **Theorem (Cunningham, 1960).** For a BS space X (1) – (3) hold:

(1) $\mathbb{P}_L(X)$ is a complete (Badé complete) Boolean algebra.

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Some Definitions

- **Notation.** For a nonempty $A \subset X$ and nonzero $\pi \in \mathbb{B}$ denote:

  $\delta_\pi(A) = \sup\{\|\pi(a-b)\| : a, b \in A\}$ (π-diameter of A).

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- **Proposition.** It is easily seen that $\delta_\pi(A) \le 2r_\pi(A)$.
- **Definition.** If $\delta_\pi(A) = 2r_\pi(A)$ for all $\pi \in \mathbb{B}$, then A is said to be $\mathbb{B}$-centerable.
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  $r_\pi(A, x) = \sup_{a \in A} \|\pi(x-a)\|$ (for all $x \in X$).
- **Proposition.** It is easily seen that $\delta_\pi(A) \leq 2r_\pi(A)$.
- **Definition.** If $\delta_\pi(A) = 2r_\pi(A)$ for all $\pi \in \mathbb{B}$, then A is said to be $\mathbb{B}$-centerable.
- **Remark.** If X' is an IBL then $\mathbb{B} = \Sigma/\mu^{-1}(0)$ and $L^1(\mathbb{B}, \mu) := L^1(\Omega, \Sigma, \mu)$ for some (Ω, Σ, μ).
- **Definition.** Denote $\mathbb{B} := \mathbb{P}_L(X)$ and take $a \in X$ and $r \in L^1(\mathbb{B}, \mu)$. A $\mathbb{B}$-cell in X is a set of the form

  $$B(a, r) := \{x \in X : \ \|\pi(x-a)\| \leq \|\pi r\|_L \text{ for all } \pi \in \mathbb{B}\}.$$

Theorem. For a real Banach space X the following are equivalent:

- (1) X' is an injective Banach lattice with $\mathbb{B} := \mathbb{P}_M(X') \simeq \mathbb{P}_L(X)$.
- (2) Every collection of four mutually intersecting $\mathbb{B}$-cells in X has nonempty intersection.
- (3) Every collection of four mutually intersecting $\mathbb{B}$-cells in X whose centers span a $(\mathbb{B}, 2)$-dimensional subspace has nonempty intersection.

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Characterization via Intersection Property

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Theorem. For a real Banach space X the following are equivalent:

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- (4) Every $\mathbb{B}$-bounded mix-compact subset of X is $\mathbb{B}$-centerable.
- (5) For every mix-compact subset A of X there exists a partition of unity $(\pi_\xi)_{\xi \in \Xi}$ in $\mathbb{B}$ such that $\pi_\xi A$ is $\pi_\xi\mathbb{B}$-centerable in $\pi_\xi X$ for all $\xi \in \Xi$.

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THANK YOU FOR ATTENTION